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Layer distortions induced by a magnetic field in planar samples of smectic C liquid crystals

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Layer distortions induced by a magnetic field in planar samples of smectic C liquid crystals

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This article derives theoretical results for the onset of the Helfrich–Hurault transition in smectic C liquid crystals induced by a magnetic field applied parallel to the smectic layers. A suitable quadratic energy in terms of the smectic layer displacement u is derived from the nonlinear version of the smectic C energy. This energy is minimized via averaging to enable the calculation of a critical field strength H_c for the onset of layer distortions. Comparisons are made with known results for the corresponding geometry in the smectic A case. An estimate for the value of the smectic C elastic constant A_{12} can also be made by considering characteristic length scales.

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1. Introduction

This article develops a model for the onset of layer distortions or undulations in smectic C (SmC) liquid crystals subjected to an applied magnetic field. There are well known theoretical results which determine a critical magnetic field magnitude H_c for the onset of layer undulations in infinite samples of smectic A (SmA) where the transition from uniformly aligned planar layers to undulated layers is known as the Helfrich–Hurault transition. The construction of a relevant energy for SmC in terms of the layer displacement u is carried out and developed here in order to extend these results from the SmA case to SmC. A major part of this work is to explore a possible energy formulation which is suitable for the analysis and determination of H_c in terms of the SmC elastic constants and other physical parameters. Comparisons with other smectic energies are also made and critical H_c values are calculated in special cases. Attention will be focused on planar layer arrangements of SmC liquid crystals; some preliminary results from an initial investigation in this context have been published by Stewart [1] and it is the intention here to present a more comprehensive account of this theory and its application. The key result for SmC is the value of H_c given by equation (78) later. This result, which does not take fully general boundary conditions into account, is particularly relevant to samples of relatively large thickness. Nevertheless, the general results obtained here are expected to be qualitatively representational of any anticipated solutions. The approach that will be implemented may be extended to cover SmC layer distortions in other geometrical alignments of the SmC layers, such as concentric cylindrical alignments of SmC [2]: such an arrangement was considered by Stewart [3] for SmA liquid crystals in a 'wedge' geometry. The motivation for the study of layer distortions in SmC and SmA comes from the work of Helfrich [4] and Hurault [5] who examined infinite samples of cholesteric liquid crystals under the influence of magnetic fields. The derivation of critical field magnitudes for infinite samples of SmA can be found in the books by de Gennes and Prost [6] and Chandrasekhar [7].

Liquid crystals are anisotropic fluids consisting of elongated molecules whose average molecular axes locally align along a common direction in space which is usually denoted by the unit vector **n**, called the director. We shall consider SmC liquid crystals which are known to form equidistant parallel layers in which **n** generally makes a fixed constant angle θ (known as the smectic tilt angle) with respect to the layer normal. The SmA phase occurs when $\theta \equiv 0$. Following de Gennes and Prost [6], SmC can be described by introducing the unit layer normal **a** and a vector **c** which is the unit orthogonal projection of **n** onto the smectic planes (see figure 1). The director **n** is related to **a** and **c**

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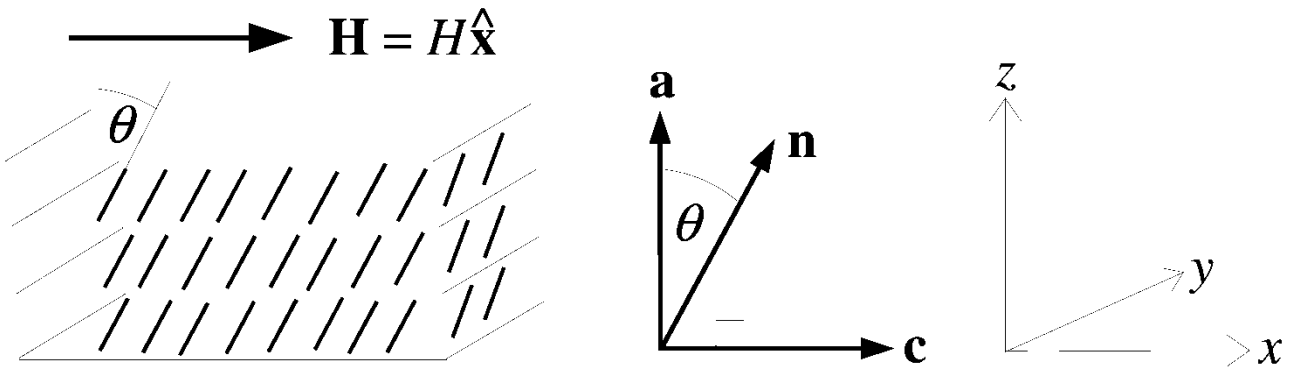

Figure 1. The arrangement of a planar aligned sample of smectic C liquid crystal described in cartesian coordinates. The director $\mathbf{n}$ is tilted at an angle θ to the layer normal $\mathbf{a}$; $\mathbf{c}$ is the unit orthogonal projection of $\mathbf{n}$ onto the smectic planes. $\mathbf{H}$ is the magnetic field.

via the equation

$$\mathbf{n} = \mathbf{a}\cos\theta + \mathbf{c}\sin\theta. \tag{1}$$

From their definitions, the vectors $\mathbf{a}$ and $\mathbf{c}$ must satisfy the constraints

$$\mathbf{a}\cdot\mathbf{a} = \mathbf{c}\cdot\mathbf{c} = 1, \quad \mathbf{a}\cdot\mathbf{c} = 0 \tag{2}$$

since these vectors are clearly unit and orthogonal. It is also mathematically convenient to introduce the unit vector $\mathbf{b}$ defined by

$$\mathbf{b} = \mathbf{a}\times\mathbf{c}. \tag{3}$$

In the absence of dislocations the layer normal $\mathbf{a}$ is subject to the additional constraint

$$\nabla\times\mathbf{a} = \mathbf{0} \tag{4}$$

originating from the work on smectics by Oseen [8]. This constraint is known to restrict the available types of equilibrium structures for smectics consisting of undistorted parallel layers which form planes, concentric cylinders, spheres and the more complex structures consisting of concentric circular tori, Dupin cyclides [6, 9–11] and parabolic cyclides [12–15]. It is expected therefore that when layer undulations or distortions occur then the above constraint in equation (4) will be broken. The construction of a suitable energy which incorporates small layer distortions is based upon relaxing the constraints (2) and (4) and examining the consequent changes to the usual (undistorted layer) SmC bulk energy. This approach has been employed in the planar aligned SmA phase in references [6, 7] and in a more general setting for SmA by Kleman and Parodi [16], whose methods we adapt for our purposes in the next section.

The nine term bulk elastic energy integrand w_b for a non-chiral SmC liquid crystal can be written in terms of the derivatives of $\mathbf{a}$ and $\mathbf{c}$ as [17]

$$\begin{aligned} w_b = {} & \frac{1}{2}A_{21}(\nabla\cdot\mathbf{a})^2 + \frac{1}{2}B_2(\nabla\cdot\mathbf{c})^2 + \frac{1}{2}B_1(\mathbf{a}\cdot\nabla\times\mathbf{c})^2 \\ & + \frac{1}{2}B_3(\mathbf{c}\cdot\nabla\times\mathbf{c})^2 + \frac{1}{2}(2A_{11}+A_{12}+A_{21}+B_3)(\mathbf{b}\cdot\nabla\times\mathbf{c})^2 \\ & - \frac{1}{2}(2A_{11}+2A_{21}+B_3)(\nabla\cdot\mathbf{a})(\mathbf{b}\cdot\nabla\times\mathbf{c}) - B_{13}(\mathbf{a}\cdot\nabla\times\mathbf{c}) \\ & \times(\mathbf{c}\cdot\nabla\times\mathbf{c}) + (C_1+C_2-B_{13})(\nabla\cdot\mathbf{c})(\mathbf{b}\cdot\nabla\times\mathbf{c}) \\ & - C_2(\nabla\cdot\mathbf{a})(\nabla\cdot\mathbf{c}) \end{aligned} \tag{5}$$

where the nine elastic constants A_{ij}, B_i and C_i are related to those introduced by the Orsay Group [18], the minor modification being that $A_{11} = -\frac{1}{2}A_{11}^{\text{Orsay}}$ and $C_1 = -C_1^{\text{Orsay}}$. Other equivalent formulations can be found in [17] and appendix B, while a physical interpretation of the elastic constants has been discussed by Carlsson *et al.* [19]; the form stated at (A19) is particularly concise, but has the disadvantage, particularly when discussing aspects of layer distortions in the geometry of figure 1, of not containing the unit layer normal $\mathbf{a}$ explicitly in its formulation. The constants A_{12}, A_{21} and A_{11} are related to bending of the smectic layers while the constants B_1, B_2, B_3 and B_{13}, are related to the reorientation of the vector $\mathbf{c}$ within or across the smectic layers. The constants C_1 and C_2 are related to various couplings of these deformations. It is known that the elastic constants obey the inequalities [19]

$$A_{12}, A_{21}, B_1, B_2, B_3 > 0 \tag{6}$$

$$A_{12}A_{21} - A_{11}^2 > 0 \tag{7}$$

$$B_1B_3 - B_{13}^2 > 0 \tag{8}$$

$$B_2A_{12} - C_1^2 > 0 \tag{9}$$

$$B_2A_{21} - C_2^2 > 0. \tag{10}$$

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Further *a priori* inequalities have been derived by Atkin and Stewart [20], namely,

$$A_{12}+A_{21}\pm 2A_{11}>0 \tag{11}$$

$$B_1+B_3\pm 2B_{13}>0 \tag{12}$$

$$B_2+A_{12}\pm 2C_1>0 \tag{13}$$

$$B_2+A_{21}\pm 2C_2>0. \tag{14}$$

It will also be relevant later to note that the smectic tilt angle dependence of the elastic constants can be approximated for small θ by [19]

$$A_{12}=K+\bar{A}_{12}\theta^2,\ A_{21}=K+\bar{A}_{21}\theta^2,\ A_{11}=-K+\bar{A}_{11}\theta^2 \tag{15}$$

$$B_1=\bar{B}_1\theta^2,\ B_2=\bar{B}_2\theta^2,\ B_3=\bar{B}_3\theta^2 \tag{16}$$

$$B_{13}=\bar{B}_{13}\theta^3,\ C_1=\bar{C}_1\theta,\ C_2=\bar{C}_2\theta \tag{17}$$

where K, $\bar{A}_{ij}$, $\bar{B}_i$ and $\bar{C}_i$ are assumed only to be weakly temperature dependent. The elastic constant $K>0$ is the usual splay constant which arises in the SmA phase elastic energy, given by [6, 7]

$$w_{\mathrm{A}}=\frac{1}{2}K(\nabla\cdot\mathbf{n})^2. \tag{18}$$

As $\theta\to 0$, it is seen from (5) and (15) to (17) that $w_{\mathrm{b}}\to w_{\mathrm{A}}$ with $\mathbf{a}\to\mathbf{n}$, by (1). The elastic constants A_{ij}, B_i and C_i all have the dimensions of energy per unit length (dyn in cgs units). Measurements are scarce in the literature, but it is expected that these elastic constants will be of the order 10^{-7} dyn, similar to that for the Frank elastic constants of nematic liquid crystal theory [6, p. 347]. A general value for the B_i constants of $B\sim 6.4\times 10^{-7}$ dyn has been indicated from the work of Schiller and Pelzl [21]. The constant K is typically of the order 6×10^{-7} dyn [6].

The magnetic energy density, ignoring a contribution which is independent of the orientation of $\mathbf{n}$, may be written as [6, p. 287]

$$w_{\mathrm{m}}=-\frac{1}{2}\chi_{\mathrm{a}}(\mathbf{n}\cdot\mathbf{H})^2 \tag{19}$$

where $\mathbf{H}$ is the magnetic field and χ_{a} is the magnetic anisotropy of the liquid crystal, typically of the order 10^{-7} in cgs units. When $\chi_{\mathrm{a}}>0$ the director prefers to align parallel with the magnetic field; $\mathbf{n}$ will tend to align perpendicular to the field when $\chi_{\mathrm{a}}<0$. We shall assume that $\chi_{\mathrm{a}}>0$ and that $\mathbf{H}$ is applied in the x-direction as shown in figure 1. In this case, the director will be attracted by the field in such a way that the layers can be expected to distort when $H=|\mathbf{H}|$ is greater than some critical value H_{c}.

The relevant energies are derived in § 2 for small layer displacements $u(x,y,z)$ in SmC and are compared with those for the SmA case. A comparison with the work of the Orsay Group is made and it is also shown in appendix B that for each of the three usual equivalent nonlinear formulations of the bulk energy w_{b}, the same quadratic energy in terms of u arises for variables separable solutions. The selection of the forms for the perturbations to the vectors $\mathbf{a}$ and $\mathbf{c}$ are motivated by the work by Stallinga and Vertogen [22] who have given a quite general formulation of these vectors for smectic phases in terms of u and its derivatives. (We note here that u corresponds to the component denoted by u_z in [22]: this is because the components u_x and u_y, also introduced in [22], do not enter the theory.) The consequent energies are summarized in § 2.5. The possibility of elementary periodic solutions dependent upon x, y and z is considered in § 3.1 where it is shown that, for small smectic tilt angles θ, no y-dependent distortions or undulations are to be expected for the geometry depicted in figure 1 during the initial stages of any distortions. Section 3.2 then goes on to consider the special case of $u=u(x,z)$. A critical magnetic field magnitude H_{c} is derived in this case for the onset of the Helfrich–Hurault transition (equation (78) below) and its connections with results elsewhere in the literature are highlighted. An estimate for the elastic constant A_{12} is also made for the SmC phase of the liquid crystal TBBA. Section 3.3 considers and comments upon the simplified version of events when $u=u(x)$. Section 4 concludes with a brief discussion.

2. Energies

In this section we construct the relevant energies for calculations and make some comparisons with earlier work on distortion energies. It will also be seen that the energy we construct will collapse in a natural way to that for the well known case of SmA liquid crystals when the limit $\theta\to 0$ is taken in the smectic tilt angle.

2.1. *Forms for* $\mathbf{a}$ *and* $\mathbf{c}$

Smectic layers are surfaces corresponding to an equation of the form

$$\Phi(x,y,z)=constant \tag{20}$$

so that the unit layer normal can be written as $\mathbf{a}=\nabla\Phi/|\nabla\Phi|$. Let $u(x,y,z)$ be a small displacement of the layers. When $u\equiv 0$ then, in the geometry of figure 1, $\Phi=z$ and $\mathbf{a}=(0,0,1)$. If such a uniform planar alignment is disturbed by the small displacement u then we can suppose that to first order in u

$$\Phi=z-u(x,y,z). \tag{21}$$

Following Kleman and Parodi [16], we can set

$$\mathbf{a}=\frac{\nabla\Phi}{|\nabla\Phi|}=\frac{\nabla\Phi}{1-\varepsilon} \tag{22}$$

$$\varepsilon=1-|\nabla\Phi| \tag{23}$$

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where ε is the relative small dilation of the layers. The expression for the SmC layer compression energy can now be derived by adapting the relevant arguments contained in [16]. Notice that when $\varepsilon=0$, or, equivalently, $|\nabla\Phi|=1$, the constraint $\nabla\times\mathbf{a}=\mathbf{0}$ in equation (4) is recovered.

In the general derivation of the bulk elastic energy there is a need to retain up to second order in u and its first derivatives when working with $\mathbf{a}$ and $\mathbf{c}$ to guarantee that all the correct terms to second order appear in the final (quadratic) energy: this is especially true for geometries other than planar, as is the case in the work of Kleman and Parodi [16] for SmA. Working to first order in u is sufficient when constructing $\mathbf{a}$ in the planar aligned SmA case, but in SmC more care is needed in, for example, the construction of the magnetic energy density because of the additional symmetries that are intrinsic to this phase. Further, this approach may well lend itself to adoption in later work for SmC in non-planar geometries.

Working to second order in u and its first derivatives gives

$$(\nabla\Phi)^2=1-2u_z+(\nabla u)^2 \tag{24}$$

$$|\nabla\Phi|=1-u_z+\frac{1}{2}\left(u_x^2+u_y^2\right) \tag{25}$$

$$|\nabla\Phi|^{-1}=1+u_z-\frac{1}{2}\left(u_x^2+u_y^2\right)+u_z^2 \tag{26}$$

where suffices denote partial differentiation of u with respect to the indicated variables. Therefore from equations (22) and (23)

$$\begin{aligned}\mathbf{a}&=\nabla\Phi|\nabla\Phi|^{-1}\\&=\left(-u_x(1+u_z),\ -u_y(1+u_z),\ 1-\frac{1}{2}\left(u_x^2+u_y^2\right)\right)\end{aligned} \tag{27}$$

$$\varepsilon=u_z-\frac{1}{2}\left(u_x^2+u_y^2\right). \tag{28}$$

This setting of $\mathbf{a}$ coincides with that of Stallinga and Vertogen [22] up to second order in the first derivatives of u. Notice, by equation (23), that ε remains as stated in (28) under the transformation $\Phi\mapsto-\Phi$; also note that u_z is unchanged under the simultaneous changes $u\mapsto-u$ and $z\mapsto-z$.

It now remains to identify a correct meaningful form for the vector $\mathbf{c}$. To this end, we adopt the general description for $\mathbf{a}$ and $\mathbf{c}$ proposed by Stallinga and Vertogen [22]. Let α, ϕ and ψ be the usual Eulerian angles where, in this present context, we let $\mathbf{a}$ be directed along the axis OC pictured in figure 2. The nutation angle $Z\hat{O}C$ is represented by α, ϕ is the precession angle $X\hat{O}N$, and ψ, being the angle $N\hat{O}A$, corresponds to a rotation of $\mathbf{a}$ about its own axis. In

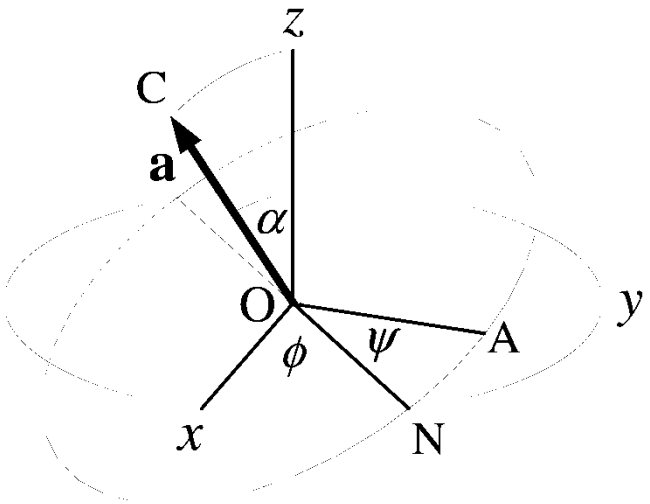

Figure 2. Description of the Eulerian angles α, ϕ and ψ relative to the usual Cartesian coordinate system used to describe the orientation of the unit vectors $\mathbf{a}$ and $\mathbf{c}$ introduced in the text.

this description we can write [22]

$$\mathbf{a}=(\sin\alpha\cos\phi,\sin\alpha\sin\phi,\cos\alpha) \tag{29}$$

$$\begin{aligned}\mathbf{c}=&-\sin(\psi)(\cos\alpha\cos\phi,\cos\alpha\sin\phi,-\sin\alpha)\\&+\cos(\psi)(-\sin\phi,\cos\phi,0)\end{aligned} \tag{30}$$

where

$$\sin\alpha=\frac{|\nabla_\perp\Phi|}{|\nabla\Phi|},\qquad \cos\alpha=\frac{1-u_z}{|\nabla\Phi|} \tag{31}$$

$$\sin\phi=-\frac{u_y}{|\nabla_\perp\Phi|},\qquad \cos\phi=-\frac{u_x}{|\nabla_\perp\Phi|} \tag{32}$$

and $\nabla_\perp\Phi=(-u_x,-u_y,0)$. There is an element of indeterminacy in the selection of ϕ and ψ in the limit as $\alpha\to0$: this is resolved by setting ϕ and ψ in such a way that $\mathbf{c}$ coincides with $(1,0,0)$ in the Cartesian coordinates to match the unperturbed configuration in figure 1 when $\alpha\equiv0$ and $u\equiv0$. By setting

$$\alpha=0\qquad\text{and}\qquad\phi+\psi=3\pi/2 \tag{33}$$

we can recover the unperturbed configuration in figure 1. Further, for small changes in α which will be related to small changes in u, it is possible to employ the approximations

$$\sin\psi=-\cos\phi+\tan\alpha\sin^2\phi \tag{34}$$

$$\cos\psi=-\cos\alpha\sin\phi-\sin\alpha\sin\phi\cos\phi \tag{35}$$

so that the conditions in equation (33) hold in the limit as $\alpha\to0$. It is easily verified that $\sin^2\psi+\cos^2\psi=1$ to second order in α when $\alpha\neq0$. Inserting equations (34) and (35) into the expression (30) and using (31) and (32) allows $\mathbf{c}$ to be evaluated to second order in the first

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derivatives of u as

$$\mathbf{c}=\left(1-\frac{1}{2}\left(u_x^2+u_y^2\right),\, u_y(1+u_z),\, u_x(1+u_z)+u_y^2\right). \quad (36)$$

Notice that $\mathbf{a}$, given by (29), coincides with the expression (27) when evaluated to second order in the first derivatives of u, which demonstrates the consistency of the above choices for ϕ and ψ in the description of $\mathbf{a}$ and $\mathbf{c}$ introduced above. The vector $\mathbf{b}=\mathbf{a}\times\mathbf{c}$ is now represented by

$$\mathbf{b}=\left(-u_y(1+u_x+u_z),\, 1-u_y^2,\, u_y(1-u_x+u_z)\right). \quad (37)$$

It is a simple exercise to verify that the SmC constraints in equation (2) are then satisfied to second order in u for these forms of $\mathbf{a}$ and $\mathbf{c}$. Additionally, we have $\mathbf{b}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{c}=0$ and $|\mathbf{b}|=1$, to second order in u. The quantity $\nabla\times\mathbf{a}$ is small and does not vanish; as mentioned in the introduction, this is as anticipated and is analogous to the situation in [16].

The form of $\mathbf{c}$ in equation (36) may be interpreted via the following observation. Given the geometry of figure 1, the director appears more likely to align further with the applied magnetic field along the x-axis rather than the y-axis in the initial stages of reorientation when $\chi_a>0$. Looking locally in the xz-plane in figure 1, an initial small perturbation in $\mathbf{c}$ ought to be like $(1,0,u_x)$ when $\mathbf{a}\approx(-u_x,0,1)$ if y-dependence is ignored and u_x is negative: this is indeed the case when the above forms for $\mathbf{a}$ and $\mathbf{c}$ are taken to first order in the first derivatives of u.

2.2. *Layer compression energy*

In the absence of deformations, the smectic layers are expected to have equal layer spacing and in these circumstances, for the present discussion on energies, we suppose initially that $\mathbf{a}=\nabla\Phi$ so that before distortions take place $\nabla\times\mathbf{a}=\mathbf{0}$ and $|\nabla\Phi|=1$. If Φ changes slightly at the onset of deformations and we assume that the density remains constant then we expect a contribution to a layer compression energy of the form [16, p.673]

$$w_{\mathrm{L}}=\frac{1}{2}(\nabla\Phi-\mathbf{a})\cdot\mathbf{B}\cdot(\nabla\Phi-a) \quad (38)$$

where the second order tensor $\mathbf{B}$ has axial symmetry around $\mathbf{a}$, due to the planar layer alignment. Thus in cartesian component form we can consider

$$B_{ij}=B_\perp\delta_{ij}+\left(B_\parallel-B_\perp\right)a_ia_j \quad (39)$$

where $B_\parallel$ and $B_\perp$ are the layer compression constants relative to the directions parallel and perpendicular to $\mathbf{a}$, respectively. The above form for $\mathbf{B}$ has been derived with the expectation that the layer normal $\mathbf{a}$ in SmC will play a similar rôle to that of the director $\mathbf{n}$ in SmA. Assuming the form for $\mathbf{a}$ given by equation (22), inserting it into (38) and (39) and simplifying gives

$$w_{\mathrm{L}}=\frac{1}{2}\frac{|\nabla\Phi|^2}{(1-\varepsilon)^2}\left[B_\perp+\left(B_\parallel-B_\perp\right)\frac{|\nabla\Phi|^2}{(1-\varepsilon)^2}\right]\varepsilon^2 \quad (40)$$

and, since equation (22) holds with $|\mathbf{a}|=1$, this finally leads to the usual type of energy term, namely,

$$w_{\mathrm{L}}=\frac{1}{2}\bar{B}\varepsilon^2 \quad (41)$$

where, for notational convenience, we henceforth denote $B_\parallel$ by $\bar{B}$. From (41) and (28) we arrive at the form

$$w_{\mathrm{L}}=\frac{1}{2}\bar{B}u_z^2 \quad (42)$$

when we retain terms to second order only in the first derivatives of u. The possible inclusion of higher order terms in w_{L} is briefly mentioned in §4. It is not necessary to include these higher order terms to determine critical thresholds; however, a description for post-threshold behaviour may require them. The layer compression constant $\bar{B}$ describes the elastic resistance to changes in the smectic layer thickness and it has dimensions of energy per unit volume (dyn cm^{-2} in cgs units): it has been estimated to be of the order 10^7 dyn cm^{-2} in SmA, for example [16, 23]. Measurements by Collin *et al.* [24] indicate that $\bar{B}\sim 8.95\times10^8$ dyn cm^{-2} for the liquid crystal TBBA in its SmA phase, while in its SmC phase $\bar{B}\sim 8.47\times10^7$ dyn cm^{-2}.

2.3. *Bulk elastic energy*

Straightforward calculations involving the above versions of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ lead to the quantities listed in appendix A. When these are substituted into the bulk energy w_{b} in equation (5) we obtain

$$\begin{aligned} w_{\mathrm{b}}=&\frac{1}{2}A_{12}u_{xx}^2+\frac{1}{2}(B_2+A_{21}+2C_2)u_{yy}^2+\frac{1}{2}B_2u_{xz}^2+\frac{1}{2}B_3u_{yz}^2\\ &+\frac{1}{2}(B_1+B_3-2B_{13})u_{xy}^2+\frac{1}{2}(2B_{13}-B_3-2A_{11}-2C_1)u_{xx}u_{yy}\\ &+(B_2+C_2)u_{yy}u_{xz}+(B_{13}-B_3)u_{xy}u_{yz}+(B_{13}-C_1)u_{xx}u_{xz}. \end{aligned} \quad (43)$$

The integrand expressions in w_{b} can be simplified following the methods outlined in [6, p. 343]. Integrating by parts with respect to x and again with respect to z shows that for any volume Ω

$$\int_\Omega u_{xz}^2\,\mathrm{d}\Omega=\int_\Omega u_{xx}u_{zz}\,\mathrm{d}\Omega+S_1 \quad (44)$$

where S_1 is a surface contribution to the total energy, in the sense that it is a quantity evaluated at some fixed

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surface. Similarly

$$\int_\Omega u_{yz}^2 \, d\Omega = \int_\Omega u_{yy} u_{zz} \, d\Omega + S_2 \tag{45}$$

$$\int_\Omega u_{xy}^2 \, d\Omega = \int_\Omega u_{xx} u_{yy} \, d\Omega + S_3 \tag{46}$$

$$\int_\Omega u_{xy} u_{yz} \, d\Omega = \int_\Omega u_{xz} u_{yy} \, d\Omega + S_4. \tag{47}$$

Terms which only ever enter the energy when evaluated at a boundary surface can be considered as not influencing the bulk layer orientation: their presence, as noted in [16], would merely shift the total energy by a constant amount, of little physical significance whenever the boundary conditions are neglected. As in the SmA case, terms involving u_{zz} can be ignored since they will be dominated by the $\bar{B}u_z^2$ term in w_L [6, p. 343]. An application of the formulae in equations (44) to (47) to w_b, omitting the u_{zz} terms and neglecting surface contributions, gives

$$\begin{aligned} w_b = {} & \frac{1}{2} A_{12} u_{xx}^2 + \frac{1}{2}(B_2 + A_{21} + 2C_2) u_{yy}^2 \\ & + \frac{1}{2}[B_1 - 2(A_{11} + C_1)] u_{xx} u_{yy} \\ & + (B_2 - B_3 + B_{13} + C_2) u_{yy} u_{xz} + (B_{13} - C_1) u_{xx} u_{xz}. \end{aligned} \tag{48}$$

Other simplifications are possible. For example, if the energy is to be invariant to the change in sign $z \to -z$ (as in the SmA case) then the last two terms in (48) can also be ignored. Rather than impose such a restriction at this stage, it is readily observed that if variables separable solutions of the form

$$u = f(x) g(y) h(z) \tag{49}$$

are sought then the last two terms in (48) integrate to products of functions evaluated on appropriate boundaries and can then be ignored for the reason mentioned above, especially if the boundaries in the xy-plane are considered to have negligible influence. In particular, if it is supposed that $0 \le z \le d$, where d is the sample depth in the z-direction, and that u has no displacement at the boundary surfaces in z, that is,

$$h(0) = h(d) = 0 \tag{50}$$

then such terms equate to zero exactly. When this is the case, then

$$\begin{aligned} w_b = {} & \frac{1}{2} A_{12} u_{xx}^2 + \frac{1}{2}(B_2 + A_{21} + 2C_2) u_{yy}^2 \\ & + \frac{1}{2}[B_1 - 2(A_{11} + C_1)] u_{xx} u_{yy} \end{aligned} \tag{51}$$

and a comparison with the bulk energy introduced by the Orsay Group [18] is then possible. Notice that the first two terms in equation (51) are positive by equations (6) and (14). In the notation for constants adopted here, the approximate nine-term Orsay energy for planar layers of SmC corresponding to equation (5) is [18]

$$\begin{aligned} w_b = {} & \frac{1}{2} B_1 \left(\frac{\partial \Omega_z}{\partial x}\right)^2 + \frac{1}{2} B_2 \left(\frac{\partial \Omega_z}{\partial y}\right)^2 + \frac{1}{2} B_3 \left(\frac{\partial \Omega_z}{\partial z}\right)^2 + B_{13} \frac{\partial \Omega_z}{\partial z} \frac{\partial \Omega_z}{\partial x} \\ & - A_{11} \left(\frac{\partial \Omega_x}{\partial x}\right)^2 + \frac{1}{2} A_{12} \left(\frac{\partial \Omega_y}{\partial x}\right)^2 + \frac{1}{2} A_{21} \left(\frac{\partial \Omega_x}{\partial y}\right)^2 \\ & - C_1 \frac{\partial \Omega_x}{\partial x} \frac{\partial \Omega_z}{\partial x} + C_2 \frac{\partial \Omega_x}{\partial y} \frac{\partial \Omega_z}{\partial y} \end{aligned} \tag{52}$$

where $\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ is an arbitrarily small rotation applied to the original undisturbed vectors **a**, **b** and **c** being given as in figure 1. As in [6, 17, 18], setting

$$\mathbf{a} \to (0,0,1) + \mathbf{\Omega} \times (0,0,1) = (\Omega_y, -\Omega_x, 1) \tag{53}$$

$$\mathbf{c} \to (1,0,0) + \mathbf{\Omega} \times (1,0,0) = (1, \Omega_z, -\Omega_y) \tag{54}$$

shows from equations (27) and (36) that, upon considering **a** and **c** to *first* order in the first derivatives of u, the energies should be comparable if we choose

$$\Omega_x = u_y, \quad \Omega_y = -u_x, \quad \Omega_z = u_y. \tag{55}$$

In [6], Ω_z was not assigned any value in terms of the derivatives of u. If, however, we adopt the above values in (55) then equation (52) collapses to (51) under the same assumptions contained in (49) and (50), using the relations (45) to (47) and, as before, omitting the terms containing u_{zz}. This demonstrates that equation (51) can be related to the alternative Orsay version to second order in the second derivatives of u if it is additionally supposed that the rotation is restricted so that (55) holds.

It should be noted here that w_b can be written in various equivalent formulations in terms of the vectors **a**, **b** and **c** and their gradients. For each such formulation the essential result for w_b in equation (51) presented above remains valid, as shown in appendix B.

2.4. *Magnetic energy*

Putting **a** and **c**, given by equations (27) and (36), into (1) and (19) with $\mathbf{H} = (H, 0, 0)$ gives, to second order in the first derivatives of u,

$$\begin{aligned} w_m = {} & -\frac{1}{2} \chi_a H^2 \Big[\cos(2\theta) u_x^2 + \left(1 - u_y^2\right) \sin^2\theta \\ & - \sin(2\theta) u_x (1 + u_z) \Big]. \end{aligned} \tag{56}$$

The term in u_x can be obviously integrated to an evaluation on the boundaries leaving the energy as

$$\begin{aligned} w_m = {} & -\frac{1}{2} \chi_a H^2 \Big[\cos(2\theta) u_x^2 + \left(1 - u_y^2\right) \sin^2\theta \\ & - \sin(2\theta) u_x u_z \Big] \end{aligned} \tag{57}$$

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while if a solution of the form given by (49) is additionally assumed then the term in $u_x u_z$ is similarly evaluated on boundaries (being zero if (50) holds). This then leads to

$$w_{\mathrm{m}} = -\frac{1}{2}\chi_{\mathrm{a}} H^2 \left[\cos(2\theta)u_x^2 + \left(1-u_y^2\right)\sin^2\theta\right]. \quad (58)$$

Although w_{m} has an unfamiliar appearance, it is particularly instructive to draw attention to the two special cases for w_{m} when $\theta=0$ and $\theta=\frac{\pi}{2}$. For $\theta=0$ the magnetic energy density ought to collapse to that for the SmA phase, and indeed

$$w_{\mathrm{m}} = -\frac{1}{2}\chi_{\mathrm{a}} H^2 u_x^2 \quad (59)$$

in this case, exactly the form used by de Gennes and Prost [6, p. 363] when considering the geometry in figure 1 for SmA with $\chi_{\mathrm{a}}>0$. For $\theta=\frac{\pi}{2}$,

$$w_{\mathrm{m}} = \frac{1}{2}\chi_{\mathrm{a}} H^2 \left(u_x^2+u_y^2\right) - \frac{1}{2}\chi_{\mathrm{a}} H^2. \quad (60)$$

Notice that there is no minus sign in the first term on the right-hand side of (60). The last term in (60) is independent of the orientation of **n** or u and, as is common practice in liquid crystal theory, can be ignored in the construction of the energy: but (60) is precisely the form for w_{m} discussed in [6, p. 344] and Stewart [25] for the case when **H** is applied *perpendicular* to the layers in SmA (in which case χ_{a} must be negative for layer distortions to occur). In this scenario, at $\theta=\frac{\pi}{2}$, the layers can effectively be thought of as becoming SmA layers perpendicular to the field depicted in figure 1. The expression for w_{m} in equation (58) for SmC is therefore a natural and meaningful extension to that for SmA; further, it is necessary to include the second order terms in **a** and **c** above, otherwise the special cases just mentioned do not collapse to the SmA cases because an additional quadratic contribution would have been overlooked. Also, as will be mentioned in § 3.3 below, w_{m} in equation (58) yields the same bulk elastic constant component (and θ dependence) as that obtained by Kedney and Stewart [26] at the critical field magnitude for an analogous problem involving an electric field when, in a first approximation in the application of SmC theory, the layer compression contribution to the energy is neglected.

2.5. *Total energy*

To summarize, the total energy density for planar aligned SmC in the geometry of figure 1, ignoring surface contributions, is given via equations (42), (48) and (57) as

$$\begin{aligned} w &= w_{\mathrm{L}} + w_{\mathrm{m}} + w_{\mathrm{b}} \\ &= \frac{1}{2}\bar{B}u_z^2 - \frac{1}{2}\chi_{\mathrm{a}} H^2 \\ &\quad \times \left[\cos(2\theta)u_x^2 + \left(1-u_y^2\right)\sin^2\theta - \sin(2\theta)u_x u_z\right] \\ &\quad + \frac{1}{2}A_{12}u_{xx}^2 + \frac{1}{2}(B_2+A_{21}+2C_2)u_{yy}^2 \\ &\quad + \frac{1}{2}[B_1 - 2(A_{11}+C_1)]u_{xx}u_{yy} \\ &\quad + (B_2 - B_3 + B_{13} + C_2)u_{yy}u_{xz} + (B_{13}-C_1)u_{xx}u_{xz}. \end{aligned} \quad (61)$$

This energy density incorporates contributions up to squared order in the derivatives of u. Moreover, if solutions of the form (49) satisfying (50) are also supposed then this energy simplifies further via equations (51) and (58) to

$$\begin{aligned} w &= \frac{1}{2}\bar{B}u_z^2 - \frac{1}{2}\chi_{\mathrm{a}} H^2 \left[\cos(2\theta)u_x^2 + \left(1-u_y^2\right)\sin^2\theta\right] \\ &\quad + \frac{1}{2}A_{12}u_{xx}^2 + \frac{1}{2}(B_2+A_{21}+2C_2)u_{yy}^2 \\ &\quad + \frac{1}{2}[B_1 - 2(A_{11}+C_1)]u_{xx}u_{yy}. \end{aligned} \quad (62)$$

The total energy integral is of course

$$W = \int_{\Omega} w \, \mathrm{d}\Omega \quad (63)$$

where Ω is the volume of the liquid crystal sample.

The results from equations (15) to (17) can be inserted into w in (61) or (62) to find that in the limit as $\theta\to 0$ we obtain the related SmA bulk energy, namely,

$$w_{\mathrm{SmA}} = \frac{1}{2}\bar{B}u_z^2 - \frac{1}{2}\chi_{\mathrm{a}} H^2 u_x^2 + \frac{1}{2}K\left(u_{xx}+u_{yy}\right)^2 \quad (64)$$

which is discussed in [6, 7, 25]. Equation (64) is also related to the energy discussed by de Gennes [27] when changes in density are ignored (for the magnetic field direction considered here). This further demonstrates that the total energy density w constructed here is a meaningful extension to SmC from the SmA theory for layer distortions.

We aim to employ periodic solutions which will be averaged over the sample volume with the purpose of determining critical behaviour via a minimization process: this is the elementary approach adopted in [6, 7].

3. Periodic solutions

We begin by showing in § 3.1 that solutions with non-zero finite periods in all three variables x, y, and z cannot minimize the averaged energy in equation (62) when the tilt angle θ is small; that is, when the sample is close to the SmA phase. This preliminary observation

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requires applications of the inequalities in §1 for the elastic constants, and is valid when considering a sample in which the boundaries to the xy-plane have a negligible effect upon the bulk of the sample. It also supports the assumption that, in general, solutions are not expected to have a periodic dependence in y for the geometry in figure 1 in the initial stages of deformations when H is near some critical threshold, and leads on to the results in §3.2 for solutions periodic in x and z.

3.1. *Solutions periodic in x, y and z*

We seek periodic solutions of the form

$$u = u_0 \sin(kx) \sin(py) \sin(q\pi z/d), \quad q \text{ a positive integer} \tag{65}$$

with u_0 a constant, which satisfy equations (49) and (50). For periodic functions f having period P, we introduce the average $\langle f \rangle$ defined by

$$\langle f \rangle = \frac{1}{P}\int_0^P f(m)\,\mathrm{d}m. \tag{66}$$

For example, any functions of the form $\sin^2(\psi)$, $\cos^2(\psi)$ or $G = constant$ result in

$$\langle \sin^2 \rangle = \langle \cos^2 \rangle = \frac{1}{2}, \qquad \langle G \rangle = G. \tag{67}$$

Since equation (65) is of variables separable form we can average the bulk energy w in (62) and employ the results from (67) to show that

$$\begin{aligned}\langle w \rangle = \frac{1}{16}u_0^2\{&\bar{B}q^2\pi^2/d^2 - \chi_a H^2\left[k^2\cos(2\theta) - p^2\sin^2(\theta)\right] \\ &+ A_{12}k^4 + (B_2 + A_{21} + 2C_2)p^4 + [B_1 - 2(A_{11} + C_1)]k^2p^2\} \\ &- \frac{1}{2}\chi_a H^2 \sin^2(\theta).\end{aligned} \tag{68}$$

This averaged energy is minimized with respect to q at $q = 1$, given that q is a positive integer. We now have to find positive values for k and p that minimize $\langle w \rangle$, given that no boundary conditions have been imposed in the x- or y-directions. For $\langle w \rangle$ to be minimized with respect to a non-zero value of p it is a necessary condition that

$$\begin{aligned}0 = \frac{\partial}{\partial p}\langle w \rangle = \frac{1}{8}u_0^2 p\{&2(B_2 + A_{21} + 2C_2)p^2 \\ &+ [B_1 - 2(A_{11} + C_1)]k^2 + \chi_a H^2 \sin^2(\theta)\}.\end{aligned} \tag{69}$$

From the inequality (14) the coefficient of p^2 inside the square bracket in equation (69) is always positive. Further, for small θ, the approximations in (15), (16) and (17) can be employed to see that

$$B_1 - 2(A_{11} + C_1) \approx \left(\bar{B}_1 - 2\bar{A}_{11}\right)\theta^2 + 2\left(K - \bar{C}_1\theta\right) > 0 \tag{70}$$

for θ sufficiently small, since $K > 0$. Thus, since $\chi_a > 0$, all the terms in the square bracket on the right-hand side of equation (69) are always positive for non-zero values of p when θ is small, and so there can be no solutions of the form (65) which are periodic in y for the problem considered here when the SmC sample is close to the SmA phase. This has the consequence that one-dimensional distortion patterns or layer undulations are to be expected rather than two-dimensional patterns. In SmA both types of pattern can occur, as indicated by Fukuda and Onuki [28] and Stewart [25], but the preliminary result above seems to indicate that one-dimensional patterns are to be preferred during the initial occurrence of distortions in the SmC phase when the smectic tilt angle θ is small.

3.2. *Solutions periodic in x and z*

When $u = u(x,z)$ the energy density for variables separable solutions in equation (62) becomes

$$w = \frac{1}{2}\bar{B}u_z^2 - \frac{1}{2}\chi_a H^2\left[\cos(2\theta)u_x^2 + \sin^2(\theta)\right] + \frac{1}{2}A_{12}u_{xx}^2. \tag{71}$$

Bearing in mind that $\langle w \rangle$ above is minimized at $q = 1$ (giving the minimum non-zero average for the layer compression energy contribution), let

$$u = u_0 \sin(kx)\sin(\pi z/d). \tag{72}$$

After averaging in a similar way to that in equation (68) over a sample of unit length in y, we have

$$\begin{aligned}\langle w \rangle = &\frac{1}{8}u_0^2\left[\bar{B}\pi^2/d^2 + A_{12}k^4 - \chi_a H^2 k^2\cos(2\theta)\right] \\ &- \frac{1}{2}\chi_a H^2 \sin^2(\theta).\end{aligned} \tag{73}$$

Also, for the undistorted state $u \equiv 0$,

$$\langle w(u \equiv 0) \rangle = -\frac{1}{2}\chi_a H^2 \sin^2(\theta) \tag{74}$$

and therefore a comparison of energies between the variable and zero solutions gives

$$\begin{aligned}\Delta\langle w \rangle &= \langle w(u) \rangle - \langle w(u \equiv 0) \rangle \\ &= \frac{1}{8}u_0^2\left[\bar{B}\pi^2/d^2 + A_{12}k^4 - \chi_a H^2 k^2\cos(2\theta)\right].\end{aligned} \tag{75}$$

The above result for $\Delta\langle w \rangle$ is reminiscent of that arising in SmA [7, p. 314]. The critical field H_c is found by minimizing $\Delta\langle w \rangle$ over non-zero values of k and then determining the least value of H above which $\Delta\langle w \rangle$ will become negative, indicating the system's preference for adopting the distorted variable solution u rather than the zero solution. The right-hand side of equation (75) can be minimized with respect to non-zero values of k to find that its minimum occurs when $k = k_x$ given by

$$k_x^2 = \frac{\chi_a H^2}{2A_{12}}\cos(2\theta). \tag{76}$$

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(Recall that $A_{12}>0$ by equation (6)). For $k=k_x$ we can write

$$\Delta\langle w\rangle=\frac{1}{8}u_0^2\left[\bar{B}\frac{\pi^2}{d^2}-\frac{\chi_a^2H^4}{4A_{12}}\cos^2(2\theta)\right]. \tag{77}$$

As H increases from zero, it is seen that the averaged energy difference $\Delta\langle w\rangle$ will decrease through zero and that the critical magnetic field strength H_c will be given by

$$\chi_a H_c^2\cos(2\theta)=2(\bar{B}A_{12})^{\frac{1}{2}}\frac{\pi}{d}\equiv 2\pi\frac{A_{12}}{\lambda d}, \tag{78}$$

where

$$\lambda=\left(\frac{A_{12}}{\bar{B}}\right)^{\frac{1}{2}} \tag{79}$$

can be introduced as a characteristic length scale to allow comparisons with known results for SmA. Additionally, k_x defined in (76), becomes k_c at $H=H_c$, where

$$k_c^2=\frac{\pi}{\lambda d}. \tag{80}$$

The product $\bar{B}A_{12}$ can be determined at H_c via equation (78).

The results from equations (15) and (16) can be inserted into (78) and (79) to find that at $\theta=0$ we have

$$\chi_a H_c^2=2\pi\frac{K}{\lambda d},\qquad \lambda=\left(\frac{K}{\bar{B}}\right)^{\frac{1}{2}} \tag{81}$$

which is precisely the result mentioned by de Gennes and Prost [6, p.363] for the critical field strength at the onset of the Helfrich–Hurault transition in SmA, K being the usual splay constant arising in (18) for the SmA elastic energy. This shows that the threshold derived in equation (78) is a natural extension of the result from the SmA case to SmC.

Following de Gennes and Prost [6, p. 363], we can estimate λ to be of the order of a microscopic length scale and, as an example, set $\lambda=20$ Å. This allows A_{12} to be estimated from equation (79) if $\bar{B}$ is known. To obtain an estimate for a typical SmC material we can take the value of $\bar{B}$ stated above in § 2.2 for the liquid crystal TBBA in its SmC phase, namely $\bar{B}=8.47\times10^7$ dyn cm^{-2}, to find from (79) that

$$A_{12}\approx 3.39\times10^{-6}\text{ dyn} \tag{82}$$

comparable to the Frank elastic constant $K=K_1$ of nematic theory which is also used in the description of SmA. We can adopt the values used in [6, p. 363] for SmA and estimate a typical critical threshold H_c for SmC by setting $\lambda=20$ Å, $\theta=22°$, sample depth $d=1$ mm, $\chi_a=10^{-7}$, and use the above estimate in (82) for A_{12}, to find that the critical value for the field strength is given by $H_c\approx122\times10^3$ Oe, which is about twice the magnitude of that for a typical SmA [6]. For example, when the other parameters are as previously stated, we can estimate $K=10^{-6}$ dyn to find that from equation $(81)_1$ we have $H_c\approx56\times10^3$ Oe for SmA.

3.3. *Solutions for $u=u(x)$*

For $u=u(x)$ the layer compression term does not enter the energy. Nevertheless, it is worthwhile seeking such solutions to compare with an earlier result by Kedney and Stewart [26] which ignored the layer compression effects in a first attempt to apply the SmC continuum theory of Leslie *et al.* [15, 17] to layer undulations induced by an electric field. Suppose the liquid crystal sample is bounded by plates at $x=0$ and $x=d$ giving the 'bookshelf' geometry (cf. figure 1) and that the layers do not distort at the boundaries. As an approximation, we can choose the first mode approach for an ansatz and set

$$u=u_0\sin\left(\frac{\pi}{d}x\right) \tag{83}$$

which ensures that $u=0$ at $x=0$, d. Over a sample volume Ω which is unit in the y and z directions with $0\le x\le d$, the energy comparison for W, given by equations (61) and (63), between u and the zero solution becomes

$$\begin{aligned}\Delta W&=W(u)-W(u\equiv0)\\&=\frac{1}{2}u_0^2\int_0^d\left[A_{12}\left(\frac{\pi}{d}\right)^4\sin^2\left(\frac{\pi}{d}x\right)\right.\\&\quad\left.-\chi_a H^2\left(\frac{\pi}{d}\right)^2\cos(2\theta)\cos^2\left(\frac{\pi}{d}x\right)\right]\mathrm{d}x\\&=\frac{1}{4}\frac{\pi^2}{d}u_0^2\left[A_{12}\left(\frac{\pi}{d}\right)^2-\chi_a H^2\cos(2\theta)\right]\end{aligned} \tag{84}$$

and therefore

$$\chi_a H_c^2\cos(2\theta)=A_{12}\left(\frac{\pi}{d}\right)^2. \tag{85}$$

This is identical to the threshold obtained by Kedney and Stewart [26, eqn. (76)] when the substitutions $\varepsilon_a\varepsilon_0\mapsto\chi_a$ and $2h\mapsto d$ are made. Their approach differs from that used here and does not include any layer compression term, just as in this special case where assuming $u=u(x)$ excludes the possibility of using the layer compression energy. Therefore the results of the previous section extend earlier ideas by incorporating layer compression or dilation for SmC and are compatible with the basic results in [26].

4. Discussion

The results presented here are consistent extensions to SmC liquid crystals of the results which are known

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for the SmA phase. As indicated above, for example, as the smectic tilt angle θ tends to zero we recover the known results for the onset of smectic layer distortions in SmA. The main result is an identification of a critical field strength H_c, given by equation (78), that indicates the possible onset of one-dimensional distortions or undulations of the smectic layer structure of the SmC phase. This critical field strength also enables an estimate to be made for the elastic constant A_{12} when $\bar{B}$ has been evaluated, or *vice-versa*: equation (82) provides an estimate of A_{12} for the liquid crystal TBBA in its SmC phase using a measurement for $\bar{B}$ reported elsewhere [24]. Approximate calculations in § 3.2 revealed that the critical magnetic field strength H_c for TBBA in the SmC phase can be expected to be higher than that anticipated for a typical SmA material. These results were obtained by searching for solutions for the layer displacement u which were periodic in x and z, and looking at the difference in the averages of the total energy of the system between the perturbed state u and the unperturbed state $u \equiv 0$. The energy density used in calculations was provided by equation (62). It was also found that if $u = u(x)$ then layer compression was absent and the earlier results obtained by Kedney and Stewart [26] by a more elementary approach were recovered.

The possibility of smectic layer 'buckling' in SmA liquid crystals induced by an electric field has been reported by Geer *et al.* [29]. The analysis of the onset and behaviour of such a phenomenon as the electric field magnitude is increased involves a different approach to that presented here and entails the analysis of a nonlinear differential equation arising form the minimization of the energy in terms of the smectic layer displacement u: higher order terms in the first derivatives of u are also taken into account in the layer compression term. Essentially, the layer compression energy becomes, via equations (28) and (41),

$$w_L = \frac{1}{2}\bar{B}\varepsilon^2 = \frac{1}{2}\bar{B}\left[u_z - \frac{1}{2}\left(u_x^2 + u_y^2\right)\right]^2 \tag{86}$$

where, as pointed out by Singer [30], inclusion of the first derivative term proportional to $(u_x^2 + u_y^2)$ is essential in order to describe any possible post-threshold transition from a sinusoidal distortion to a buckling distortion, as the magnitude of the field increases above the critical field strength. The investigation of the possibility of layer buckling in SmC liquid crystals may well be worthwhile, given that similarities have been identified in § 2 and § 3 for SmA and SmC at the onset of smectic layer distortions. To obtain a more accurate model of distortions as the field magnitude increases, we would expect to include more elastic terms in the above description at equation (62) for SmC by constructing the energy to a higher order in the derivatives and powers of u. The model introduced above in § 2 and § 3 will be particularly relevant near the critical field threshold.

It should perhaps also be mentioned that Ribotta and Durand [31] have considered mechanical instabilities in SmA liquid crystals induced by dilative or compressive stresses. Their theoretical predictions and experimental results may be of interest if one were to contemplate similar experimental set-ups for SmC liquid crystals. Ribotta and Durand also considered some elementary aspects of the dynamics of an undulation instability in SmA and these may prove to be relevant in the dynamics of any possible distortions or undulations in SmC liquid crystals.

Some preliminary results for layer distortions in ferroelectric smectic C liquid crystals (SmC*) have been reported by Stewart [32] in the case of an applied electric field. The critical electric field threshold E_c is more complex than the form for H_c and involves other elastic constants and the spontaneous polarization P_0, inherent to SmC* liquid crystals.

The techniques used here may be of relevance to work on electrically driven instabilities in thin films of SmC or SmC* liquid crystals and in this context the reader should consult the work of Ried *et al.* [33, 34]. There is also a comprehensive study of electrically driven effects and electroconvection by Pleiner *et al.* [35]. Some experiments on electroconvection effects on freely suspended films of SmC and SmC* by Langer and Stannarius [36] have also been made.

In conclusion, the investigation of smectic layer distortions in SmC presented here may lead to a valuable experimental determination of the elastic constant A_{12} via the critical magnetic field strength H_c given by equation (78). The relationship between this constant and the SmC layer compression constant $\bar{B}$ may also be determined.

Appendix A

Calculations of the terms in w_b in equation (5) using the values for **a**, **b** and **c** in equations (27), (37) and (36), respectively, give the following quantities to first order in u:

$$\nabla\cdot\mathbf{a} = -\left(u_{xx} + u_{yy}\right) \tag{A1}$$

$$\nabla\cdot\mathbf{c} = u_{yy} + u_{xz} \tag{A2}$$

$$\mathbf{a}\cdot\nabla\times\boldsymbol{c} = u_{xy} \tag{A3}$$

$$\mathbf{b}\cdot\nabla\times\boldsymbol{c} = -u_{xx} \tag{A4}$$

$$\mathbf{c}\cdot\nabla\times\boldsymbol{c} = u_{xy} - u_{yz}. \tag{A5}$$

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These lead to the following quantities that are used in § 2.3 when the elastic energy density is calculated to second order in u:

$$(\nabla\cdot\mathbf{a})^2=\left(u_{xx}+u_{yy}\right)^2 \tag{A6}$$

$$(\nabla\cdot\mathbf{c})^2=\left(u_{yy}+u_{xz}\right)^2 \tag{A7}$$

$$(\mathbf{a}\cdot\nabla\times\mathbf{c})^2=u_{xy}^2 \tag{A8}$$

$$(\mathbf{b}\cdot\nabla\times\mathbf{c})^2=u_{xx}^2 \tag{A9}$$

$$(\mathbf{c}\cdot\nabla\times\mathbf{c})^2=\left(u_{xy}-u_{yz}\right)^2 \tag{A10}$$

$$(\nabla\cdot\mathbf{a})(\mathbf{b}\cdot\nabla\times\mathbf{c})=u_{xx}\left(u_{xx}+u_{yy}\right) \tag{A11}$$

$$(\mathbf{a}\cdot\nabla\times\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{c})=u_{xy}\left(u_{xy}-u_{yz}\right) \tag{A12}$$

$$(\nabla\cdot\mathbf{c})(\mathbf{b}\cdot\nabla\times\mathbf{c})=-u_{xx}\left(u_{yy}+u_{xz}\right) \tag{A13}$$

$$(\nabla\cdot\mathbf{a})(\nabla\cdot\mathbf{c})=-\left(u_{xx}+u_{yy}\right)\left(u_{yy}+u_{xz}\right). \tag{A14}$$

Appendix B

The bulk energy w_b in equation (5) has two other equivalent nonlinear formulations of general interest which are appropriately quoted below. The first includes gradients of **b** and **c** in its formulation and the second incorporates gradients in **a** and **b**. The formulation of energy is usually chosen according to the particular geometry being investigated. Clearly, w_L does not change while w_m remains the same since it is independent of the gradients of the director. It will now be verified here that w_b given by equation (62) (and hence also $w=w_\mathrm{L}+w_\mathrm{m}+w_\mathrm{b}$) is independent of the choice of the energy formulation.

For **a**, **b** and **c** given by equations (27), (37) and (36), respectively, we have, in addition to the quantities derived in appendix A, the following to first order in u:

$$\nabla\cdot\mathbf{b}=u_{yz}-u_{xy} \tag{A15}$$

$$\mathbf{a}\cdot\nabla\times\mathbf{b}=u_{yy} \tag{A16}$$

$$\mathbf{b}\cdot\nabla\times\mathbf{b}=-\left(u_{xy}+u_{yz}\right) \tag{A17}$$

$$\mathbf{c}\cdot\nabla\times\mathbf{b}=u_{yy}. \tag{A18}$$

For the **b** and **c** formulation given in [17], these quantities give, to second order in u,

$$\begin{aligned}
w_\mathrm{b}=&\frac{1}{2}A_{12}(\mathbf{b}\cdot\nabla\times\mathbf{c})^2+\frac{1}{2}A_{21}(\mathbf{c}\cdot\nabla\times\mathbf{b})^2\\
&+A_{11}(\mathbf{b}\cdot\nabla\times\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{b})+\frac{1}{2}B_1(\nabla\cdot\mathbf{b})^2+\frac{1}{2}B_2(\nabla\cdot\mathbf{c})^2\\
&+\frac{1}{2}B_3\left[\frac{1}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b}+\mathbf{c}\cdot\nabla\times\mathbf{c})\right]^2\\
&+\frac{1}{2}B_{13}(\nabla\cdot\mathbf{b})[(\mathbf{b}\cdot\nabla\times\mathbf{b}+\mathbf{c}\cdot\nabla\times\mathbf{c})]\\
&+C_1(\nabla\cdot\mathbf{c})(\mathbf{b}\cdot\nabla\times\mathbf{c})+C_2(\nabla\cdot\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{b})
\end{aligned} \tag{A19}$$

$$\begin{aligned}
=&\frac{1}{2}A_{12}u_{xx}^2+\frac{1}{2}(B_2+A_{21}+2C_2)u_{yy}^2\\
&+\frac{1}{2}(B_1+B_3-2B_{13})u_{yz}^2+\frac{1}{2}B_1u_{xy}^2-(A_{11}+C_1)u_{xx}u_{yy}\\
&+\frac{1}{2}B_2u_{xz}^2+(B_2+C_2)u_{yy}u_{xz}-C_1u_{xx}u_{xz}\\
&+(B_{13}-B_1)u_{xy}u_{yz}.
\end{aligned} \tag{A20}$$

By using the identities (44) to (47) and discarding the terms in u_{zz} and surface contributions (for the same reasons as given in the derivation of (48)), this now leads to

$$\begin{aligned}
w_\mathrm{b}=&\frac{1}{2}A_{12}u_{xx}^2+\frac{1}{2}(B_2+A_{21}+2C_2)u_{yy}^2\\
&+\frac{1}{2}[B_1-2(A_{11}+C_1)]u_{xx}u_{yy}-C_1u_{xx}u_{xz}\\
&+(B_2-B_1+B_{13}+C_2)u_{xy}u_{yz}
\end{aligned} \tag{A21}$$

which becomes precisely w_b in equation (51) when u is chosen as a variables separable solution satisfying the conditions (50), following from the comments after equation (49) above.

For the **a** and **b** formulation given in [17] we have

$$\begin{aligned}
w_\mathrm{b}=&\frac{1}{2}A_{12}(\nabla\cdot\mathbf{a})^2+\frac{1}{2}B_1(\nabla\cdot\mathbf{b})^2+\frac{1}{2}B_2(\mathbf{a}\cdot\nabla\times\mathbf{b})^2\\
&+\frac{1}{2}B_3(\mathbf{b}\cdot\nabla\times\mathbf{b})^2+\frac{1}{2}(2A_{11}+A_{12}+A_{21}+B_3)(\mathbf{c}\cdot\nabla\times\mathbf{b})^2\\
&+\frac{1}{2}(2A_{11}+2A_{12}+B_3)(\nabla\cdot\mathbf{a})(\mathbf{c}\cdot\nabla\times\mathbf{b})\\
&+B_{13}(\nabla\cdot\mathbf{b})(\mathbf{b}\cdot\nabla\times\mathbf{b})+(B_{13}+C_1)(\nabla\cdot\mathbf{a})(\mathbf{a}\cdot\nabla\times\mathbf{b})\\
&+(B_{13}+C_1+C_2)(\mathbf{a}\cdot\nabla\times\mathbf{b})(\mathbf{c}\cdot\nabla\times\mathbf{b})
\end{aligned} \tag{A22}$$

$$\begin{aligned}
=&\frac{1}{2}A_{12}u_{xx}^2+\frac{1}{2}(B_2+A_{21}+2C_2)u_{yy}^2\\
&-\frac{1}{2}(2A_{11}+B_3+2B_{13}+2C_1)u_{xx}u_{yy}\\
&+\frac{1}{2}(B_1+B_3-2B_{13})u_{yz}^2+(B_3-B_1)u_{xy}u_{yz}\\
&+\frac{1}{2}(B_1+B_3+2B_{13})u_{xy}^2.
\end{aligned} \tag{A23}$$

Again, we can invoke the identities (44) to (47) and discard terms in u_{zz} and surface contributions,

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resulting in

$$w_b = \frac{1}{2}A_{12}u_{xx}^2 + \frac{1}{2}(B_2 + A_{21} + 2C_2)u_{yy}^2 + \frac{1}{2}[B_1 - 2(A_{11} + C_1)]u_{xx}u_{yy} + (B_3 - B_1)u_{xy}u_{yz}. \quad \text{(A24)}$$

As before, the last term in the above can be neglected for variables separable solutions satisfying (50), resulting in the expression for w_b used in equation (51).

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